Chapter 4

PRINCIPLE OF MATHEMATICAL INDUCTION

4.1 Overview

Mathematical induction is one of the techniques which can be used to prove variety of mathematical statements which are formulated in terms of n, where n is a positive integer.

4.1.1 *The principle of mathematical induction*

Let P(n) be a given statement involving the natural number n such that

(i) The statement is true for $n = 1$, i.e., P(1) is true (or true for any fixed natural number) and

(ii) If the statement is true for $n = k$ (where k is a particular but arbitrary natural number), then the statement is also true for $n = k + 1$, i.e, truth of P(k) implies the truth of P(k + 1). Then P(n) is true for all natural numbers n.

4.2 Solved Examples

Short Answer Type

Prove statements in Examples 1 to 5, by using the Principle of Mathematical Induction for all $n \in \mathbf{N}$, that :

Example 1 $1 + 3 + 5 + ... + (2n - 1) = n^2$

Solution Let the given statement P(n) be defined as P(n) : $1 + 3 + 5 + ... + (2n - 1) = n^2$, for $n \in \mathbf{N}$. Note that P(1) is true, since

$$P(1) : 1 = 1^2$$

Assume that P(k) is true for some $k \in \mathbf{N}$, i.e.,

$$P(k) : 1 + 3 + 5 + ... + (2k - 1) = k^2$$

Now, to prove that P(k + 1) is true, we have

$$1 + 3 + 5 + ... + (2k - 1) + (2k + 1)$$
$$= k^2 + (2k + 1) \qquad \text{(Why?)}$$
$$= k^2 + 2k + 1 = (k + 1)^2$$

Thus, P(k + 1) is true, whenever P(k) is true.

Hence, by the Principle of Mathematical Induction, P(n) is true for all $n \in \mathbf{N}$.

Example 2 $\sum_{t=1}^{n-1} t(t+1) = \frac{n(n-1)(n+1)}{3}$, for all natural numbers $n \geq 2$.

Solution Let the given statement P(n), be given as

$$P(n): \sum_{t=1}^{n-1} t(t+1) = \frac{n(n-1)(n+1)}{3}, \text{ for all natural numbers } n \geq 2.$$

We observe that

$$P(2): \sum_{t=1}^{2-1} t(t+1) = \sum_{t=1}^{1} t(t+1) = 1.2 = \frac{1.2.3}{3}$$

$$= \frac{2.(2-1)(2+1)}{3}$$

Thus, P(n) in true for n = 2.

Assume that P(n) is true for $n = k \in \mathbf{N}$.

i.e.,
$$P(k): \sum_{t=1}^{k-1} t(t+1) = \frac{k(k-1)(k+1)}{3}$$

To prove that P(k + 1) is true, we have

$$\sum_{t=1}^{(k+1-1)} t(t+1) = \sum_{t=1}^{k} t(t+1)$$

$$= \sum_{t=1}^{k-1} t(t+1) + k(k+1) = \frac{k(k-1)(k+1)}{3} + k(k+1)$$

$$= k(k+1)\left[\frac{k-1+3}{3}\right] = \frac{k(k+1)(k+2)}{3}$$

$$= \frac{(k+1)((k+1)-1))((k+1)+1)}{3}$$

Thus, P(k + 1) is true, whenever P(k) is true.

Hence, by the Principle of Mathematical Induction, P(n) is true for all natural numbers $n \geq 2$.

Example 3 $\left(1-\frac{1}{2^2}\right).\left(1-\frac{1}{3^2}\right)...\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$, for all natural numbers, $n \geq 2$.

Solution Let the given statement be P(n), i.e.,

$$P(n) : \left(1-\frac{1}{2^2}\right).\left(1-\frac{1}{3^2}\right)...\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n}\text{, for all natural numbers, } n \geq 2$$

We, observe that P (2) is true, since

$$\left(1-\frac{1}{2^2}\right)=1-\frac{1}{4} = \frac{4-1}{4}=\frac{3}{4}=\frac{2+1}{2\times 2}$$

Assume that P(n) is true for some $k \in$ **N**, i.e.,

$$P(k) : \left(1-\frac{1}{2^2}\right).\left(1-\frac{1}{3^2}\right)...\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k}$$

Now, to prove that P (k + 1) is true, we have

$$\left(1-\frac{1}{2^2}\right).\left(1-\frac{1}{3^2}\right)...\left(1-\frac{1}{k^2}\right).\left(1-\frac{1}{(k+1)^2}\right)$$

$$=\frac{k+1}{2k}\left(1-\frac{1}{(k+1)^2}\right) = \frac{k^2+2k}{2k\,(k+1)} = \frac{(k+1)+1}{2(k+1)}$$

Thus, P (k + 1) is true, whenever P(k) is true.

Hence, by the Principle of Mathematical Induction, P(n) is true for all natural numbers, $n \geq 2$.

Example 4 $2^{2n} - 1$ is divisible by 3.

Solution Let the statement P(n) given as

P(n) : $2^{2n} - 1$ is divisible by 3, for every natural number n.

We observe that P(1) is true, since

$$2^2 - 1 = 4 - 1 = 3.1 \text{ is divisible by 3.}$$

Assume that P(n) is true for some natural number k, i.e.,

P(k): $2^{2k} - 1$ is divisible by 3, i.e., $2^{2k} - 1 = 3q$, where $q \in$ **N**

Now, to prove that P(k + 1) is true, we have

$$P(k + 1) : 2^{2(k+1)} - 1 = 2^{2k+2} - 1 = 2^{2k} . 2^2 - 1$$

$$= 2^{2k} . 4 - 1 = 3.2^{2k} + (2^{2k} - 1)$$

$$= 3.2^{2k} + 3q$$
$$= 3\,(2^{2k} + q) = 3m, \text{ where } m \in \mathbf{N}$$

Thus P(k + 1) is true, whenever P(k) is true.

Hence, by the Principle of Mathematical Induction P(n) is true for all natural numbers n.

Example 5 $2n + 1 < 2^n$, for all natual numbers $n \geq 3$.

Solution Let P(n) be the given statement, i.e., P(n) : $(2n + 1) < 2^n$ for all natural numbers, $n \geq 3$. We observe that P(3) is true, since

$$2.3 + 1 = 7 < 8 = 2^3$$

Assume that P(n) is true for some natural number k, i.e., $2k + 1 < 2^k$

To prove P(k + 1) is true, we have to show that $2(k + 1) + 1 < 2^{k+1}$. Now, we have

$$2(k + 1) + 1 = 2\,k + 3$$
$$= 2k + 1 + 2 < 2^k + 2 < 2^k\,.\,2 = 2^{k+1}.$$

Thus P(k + 1) is true, whenever P(k) is true.

Hence, by the Principle of Mathematical Induction P(n) is true for all natural numbers, $n \geq 3$.

Long Answer Type

Example 6 Define the sequence a_1, a_2, a_3... as follows :

$a_1 = 2, a_n = 5\,a_{n-1}$, for all natural numbers $n \geq 2$.

(i) Write the first four terms of the sequence.

(ii) Use the Principle of Mathematical Induction to show that the terms of the sequence satisfy the formula $a_n = 2.5^{n-1}$ for all natural numbers.

Solution

(i) We have $a_1 = 2$

$a_2 = 5a_{2-1} = 5a_1 = 5.2 = 10$

$a_3 = 5a_{3-1} = 5a_2 = 5.10 = 50$

$a_4 = 5a_{4-1} = 5a_3 = 5.50 = 250$

(ii) Let P(n) be the statement, i.e.,

P(n) : $a_n = 2.5^{\,n-1}$ for all natural numbers. We observe that P(1) is true

Assume that P(n) is true for some natural number k, i.e., P(k) : $a_k = 2.5^{k-1}$.

Now to prove that P (k + 1) is true, we have

$$P(k+1) : a_{k+1} = 5.a_k = 5 \cdot (2.5^{k-1})$$
$$= 2.5^k = 2.5^{(k+1)-1}$$

Thus P(k + 1) is true whenever P (k) is true.

Hence, by the Principle of Mathematical Induction, P(n) is true for all natural numbers.

Example 7 The distributive law from algebra says that for all real numbers c, a_1 and a_2, we have $c\,(a_1 + a_2) = ca_1 + ca_2$.

Use this law and mathematical induction to prove that, for all natural numbers, $n \geq 2$, if $c, a_1, a_2, ..., a_n$ are any real numbers, then

$$c\,(a_1 + a_2 + ... + a_n) = ca_1 + ca_2 + ... + ca_n$$

Solution Let P(n) be the given statement, i.e.,

P(n) : $c\,(a_1 + a_2 + ... + a_n) = ca_1 + ca_2 + ...\ ca_n$ for all natural numbers $n \geq 2$, for c, a_1, a_2, ... $a_n \in$ **R**.

We observe that P(2) is true since

$$c(a_1 + a_2) = ca_1 + ca_2 \qquad \text{(by distributive law)}$$

Assume that P(n) is true for some natural number k, where $k > 2$, i.e.,

$$P(k) : c\,(a_1 + a_2 + ... + a_k) = ca_1 + ca_2 + ... + ca_k$$

Now to prove P(k + 1) is true, we have

$$P(k+1) : c\,(a_1 + a_2 + ... + a_k + a_{k+1})$$
$$= c\,((a_1 + a_2 + ... + a_k) + a_{k+1})$$
$$= c\,(a_1 + a_2 + ... + a_k) + ca_{k+1} \qquad \text{(by distributive law)}$$
$$= ca_1 + ca_2 + ... + ca_k + ca_{k+1}$$

Thus P(k + 1) is true, whenever P (k) is true.

Hence, by the principle of Mathematical Induction, P(n) is true for all natural numbers $n \geq 2$.

Example 8 Prove by induction that for all natural number n

$\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + ... + \sin(\alpha + (n-1)\beta)$

$$= \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

Solution Consider P (n) : $\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + ... + \sin(\alpha + (n-1)\beta)$

$$= \frac{\sin(\alpha + \frac{n-1}{2}\beta)\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}, \text{ for all natural number } n.$$

We observe that

P (1) is true, since

$$\text{P (1)} : \sin\alpha = \frac{\sin(\alpha + 0)\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}}$$

Assume that P(n) is true for some natural numbers k, i.e.,

P (k) : $\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + ... + \sin(\alpha + (k-1)\beta)$

$$= \frac{\sin(\alpha + \frac{k-1}{2}\beta)\sin\left(\frac{k\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

Now, to prove that P (k + 1) is true, we have

P (k+ 1) : $\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + ... + \sin(\alpha + (k-1)\beta) + \sin(\alpha + k\beta)$

$$= \frac{\sin(\alpha + \frac{k-1}{2}\beta)\sin\left(\frac{k\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} + \sin(\alpha + k\beta)$$

$$= \frac{\sin\left(\alpha + \frac{k-1}{2}\beta\right)\sin\frac{k\beta}{2} + \sin(\alpha + k\beta)\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}}$$

$$= \frac{\cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + k\beta - \frac{\beta}{2}\right) + \cos\left(\alpha + k\beta - \frac{\beta}{2}\right) - \cos\left(\alpha + k\beta + \frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{\cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + k\beta + \frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{\sin\left(\alpha + \frac{k\beta}{2}\right)\sin\left(\frac{k\beta + \beta}{2}\right)}{\sin\frac{\beta}{2}}$$

$$= \frac{\sin\left(\alpha + \frac{k\beta}{2}\right)\sin\,(k+1)\left(\frac{\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

Thus P $(k + 1)$ is true whenever P (k) is true.

Hence, by the Principle of Mathematical Induction P(n) is true for all natural number n.

Example 9 Prove by the Principle of Mathematical Induction that

$1 \times 1! + 2 \times 2! + 3 \times 3! + ... + n \times n! = (n + 1)! - 1$ for all natural numbers n.

Solution Let P(n) be the given statement, that is,

P(n) : $1 \times 1! + 2 \times 2! + 3 \times 3! + ... + n \times n! = (n + 1)! - 1$ for all natural numbers n.

Note that P (1) is true, since

$$P(1) : 1 \times 1! = 1 = 2 - 1 = 2! - 1.$$

Assume that P(n) is true for some natural number k, i.e.,

P(k) : $1 \times 1! + 2 \times 2! + 3 \times 3! + ... + k \times k! = (k + 1)! - 1$

To prove P $(k + 1)$ is true, we have

P $(k + 1)$: $1 \times 1! + 2 \times 2! + 3 \times 3! + ... + k \times k! + (k + 1) \times (k + 1)!$

$$= (k + 1)! - 1 + (k + 1)! \times (k + 1)$$

$$= (k + 1 + 1)\ (k + 1)! - 1$$

$$= (k + 2)\ (k + 1)! - 1 = ((k + 2)! - 1$$

Thus P $(k + 1)$ is true, whenever P (k) is true. Therefore, by the Principle of Mathematical Induction, P (n) is true for all natural number n.

Example 10 Show by the Principle of Mathematical Induction that the sum S_n of the n term of the series $1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + 5^2 + 2 \times 6^2$ ... is given by

$$S_n = \begin{cases} \dfrac{n(n+1)^2}{2}, & \text{if } n \text{ is even} \\ \dfrac{n^2(n+1)}{2}, & \text{if } n \text{ is odd} \end{cases}$$

Solution Here $P(n) : S_n = \begin{cases} \dfrac{n(n+1)^2}{2}, & \text{when } n \text{ is even} \\ \dfrac{n^2(n+1)}{2}, & \text{when } n \text{ is odd} \end{cases}$

Also, note that any term T_n of the series is given by

$$T_n = \begin{cases} n^2 \text{ if } n \text{ is odd} \\ 2n^2 \text{ if } n \text{ is even} \end{cases}$$

We observe that P(1) is true since

$$P(1) : S_1 = 1^2 = 1 = \frac{1.2}{2} = \frac{1^2.(1+1)}{2}$$

Assume that $P(k)$ is true for some natural number k, i.e.

Case 1 When k is odd, then $k + 1$ is even. We have

$P(k + 1) : S_{k+1} = 1^2 + 2 \times 2^2 + ... + k^2 + 2 \times (k + 1)^2$

$$= \frac{k^2(k+1)}{2} + 2 \times (k + 1)^2$$

$$= \frac{(k+1)}{2} [k^2 + 4(k + 1)] \text{ (as } k \text{ is odd, } 1^2 + 2 \times 2^2 + ... + k^2 = k^2 \frac{(k+1)}{2})$$

$$= \frac{k+1}{2} [k^2 + 4k + 4]$$

$$= \frac{k+1}{2} (k+2)^2 = (k + 1) \frac{[(k+1)+1]^2}{2}$$

So $P(k + 1)$ is true, whenever $P(k)$ is true in the case when k is odd.

Case 2 When k is even, then $k + 1$ is odd.

Now, $\quad P(k+1) : 1^2 + 2 \times 2^2 + ... + 2.k^2 + (k+1)^2$

$$= \frac{k(k+1)^2}{2} + (k+1)^2 \quad \left(\text{as } k \text{ is even, } 1^2 + 2 \times 2^2 + ... + 2k^2 = k\,\frac{(k+1)^2}{2}\right)$$

$$= \frac{(k+1)^2\,(k+2)}{2} = \frac{(k+1)^2\,((k+1)+1)}{2}$$

Therefore, P $(k + 1)$ is true, whenever P (k) is true for the case when k is even. Thus P $(k + 1)$ is true whenever P (k) is true for any natural numbers k. Hence, P (n) true for all natural numbers.

Objective Type Questions

Choose the correct answer in Examples 11 and 12 (M.C.Q.)

Example 11 Let P(n) : "$2^n < (1 \times 2 \times 3 \times ... \times n)$". Then the smallest positive integer for which P (n) is true is

(A) 1 (B) 2 (C) 3 (D) 4

Solution Answer is D, since

P (1) : $2 < 1$ is false

P (2) : $2^2 < 1 \times 2$ is false

P (3) : $2^3 < 1 \times 2 \times 3$ is false

But P (4) : $2^4 < 1 \times 2 \times 3 \times 4$ is true

Example 12 A student was asked to prove a statement P (n) by induction. He proved that P $(k + 1)$ is true whenever P (k) is true for all $k > 5 \in \mathbf{N}$ and also that P (5) is true. On the basis of this he could conclude that P (n) is true

(A) for all $n \in \mathbf{N}$ (B) for all $n > 5$

(C) for all $n \geq 5$ (D) for all $n < 5$

Solution Answer is (C), since P(5) is true and P$(k + 1)$ is true, whenever P (k) is true.

Fill in the blanks in Example 13 and 14.

Example 13 If P (n) : "$2.4^{2n+1} + 3^{3n+1}$ is divisible by λ for all $n \in \mathbf{N}$" is true, then the value of λ is ____

Solution Now, for $n = 1$,

$$2.4^{2+1} + 3^{3+1} = 2.4^3 + 3^4 = 2.64 + 81 = 128 + 81 = 209,$$

for $\quad n = 2,\ 2.4^5 + 3^7 = 8.256 + 2187 = 2048 + 2187 = 4235$

Note that the H.C.F. of 209 and 4235 is 11. So $2.4^{2n+1} + 3^{3n+1}$ is divisible by 11. Hence, λ is 11

Example 14 If P (n) : "$49^n + 16^n + k$ is divisible by 64 for $n \in \mathbf{N}$" is true, then the least negative integral value of k is ______.

Solution For $n = 1$, P(1) : $65 + k$ is divisible by 64.

Thus k, should be – 1 since, $65 – 1 = 64$ is divisible by 64.

Example 15 State whether the following proof (by mathematical induction) is true or false for the statement.

$$P(n):\ 1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof By the Principle of Mathematical induction, P(n) is true for $n = 1$,

$1^2 = 1 = \dfrac{1(1+1)(2\cdot1+1)}{6}$. Again for some $k \geq 1$, $k^2 = \dfrac{k(k+1)(2k+1)}{6}$. Now we prove that

$$(k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

Solution False

Since in the inductive step both the inductive hypothesis and what is to be proved are wrong.

4.3 EXERCISE

Short Answer Type

1. Give an example of a statement P(n) which is true for all $n \geq 4$ but P(1), P(2) and P(3) are not true. Justify your answer.
2. Give an example of a statement P(n) which is true for all n. Justify your answer.

   Prove each of the statements in Exercises 3 - 16 by the Principle of Mathematical Induction :

3. $4^n – 1$ is divisible by 3, for each natural number n.
4. $2^{3n} – 1$ is divisible by 7, for all natural numbers n.
5. $n^3 – 7n + 3$ is divisible by 3, for all natural numbers n.
6. $3^{2n} – 1$ is divisible by 8, for all natural numbers n.

7. For any natural number n, $7^n - 2^n$ is divisible by 5.
8. For any natural number n, $x^n - y^n$ is divisible by $x - y$, where x and y are any integers with $x \neq y$.
9. $n^3 - n$ is divisible by 6, for each natural number $n \geq 2$.
10. $n (n^2 + 5)$ is divisible by 6, for each natural number n.
11. $n^2 < 2^n$ for all natural numbers $n \geq 5$.
12. $2n < (n + 2)!$ for all natural number n.
13. $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}}$, for all natural numbers $n \geq 2$.
14. $2 + 4 + 6 + ... + 2n = n^2 + n$ for all natural numbers n.
15. $1 + 2 + 2^2 + ... + 2^n = 2^{n+1} - 1$ for all natural numbers n.
16. $1 + 5 + 9 + ... + (4n - 3) = n (2n - 1)$ for all natural numbers n.

Long Answer Type

Use the Principle of Mathematical Induction in the following Exercises.

17. A sequence a_1, a_2, a_3 ... is defined by letting $a_1 = 3$ and $a_k = 7a_{k-1}$ for all natural numbers $k \geq 2$. Show that $a_n = 3.7^{n-1}$ for all natural numbers.
18. A sequence b_0, b_1, b_2 ... is defined by letting $b_0 = 5$ and $b_k = 4 + b_{k-1}$ for all natural numbers k. Show that $b_n = 5 + 4n$ for all natural number n using mathematical induction.
19. A sequence d_1, d_2, d_3 ... is defined by letting $d_1 = 2$ and $d_k = \frac{d_{k-1}}{k}$ for all natural numbers, $k \geq 2$. Show that $d_n = \frac{2}{n!}$ for all $n \in \mathbf{N}$.
20. Prove that for all $n \in \mathbf{N}$
$\cos \alpha + \cos (\alpha + \beta) + \cos (\alpha + 2\beta) + ... + \cos (\alpha + (n - 1) \beta)$
$$= \frac{\cos\left(\alpha + \left(\frac{n-1}{2}\right)\beta\right)\sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}}$$
21. Prove that, $\cos \theta \cos 2\theta \cos 2^2\theta ... \cos 2^{n-1}\theta = \frac{\sin 2^n\theta}{2^n \sin\theta}$, for all $n \in \mathbf{N}$.
22. Prove that, $\sin\theta + \sin 2\theta + \sin 3\theta + ... + \sin n\theta = \frac{\sin\frac{n\theta}{2}\sin\frac{(n+1)}{2}\theta}{\sin\frac{\theta}{2}}$, for all $n \in \mathbf{N}$.

23. Show that $\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}$ is a natural number for all $n \in$ **N.**

24. Prove that $\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{2n}>\frac{13}{24}$, for all natural numbers $n > 1$.

25. Prove that number of subsets of a set containing n distinct elements is 2^n, for all $n \in$ **N**.

Objective Type Questions

Choose the correct answers in Exercises 26 to 30 (M.C.Q.).

26. If $10^n + 3.4^{n+2} + k$ is divisible by 9 for all $n \in$ **N**, then the least positive integral value of k is

(A) 5 (B) 3 (C) 7 (D) 1

27. For all $n \in$ **N**, $3.5^{2n+1} + 2^{3n+1}$ is divisible by

(A) 19 (B) 17 (C) 23 (D) 25

28. If $x^n - 1$ is divisible by $x - k$, then the least positive integral value of k is

(A) 1 (B) 2 (C) 3 (D) 4

Fill in the blanks in the following :

29. If P(n) : $2n < n!$, $n \in$ **N**, then P(n) is true for all $n \geq$ ________.

State whether the following statement is true or false. Justify.

30. Let P(n) be a statement and let P(k) $\Rightarrow$ P(k + 1), for some natural number k, then P(n) is true for all $n \in$ **N**.

Chapter 5

COMPLEX NUMBERS AND QUADRATIC EQUATIONS

5.1 Overview

We know that the square of a real number is always non-negative e.g. $(4)^2 = 16$ and $(-4)^2 = 16$. Therefore, square root of 16 is ± 4. What about the square root of a negative number? It is clear that a negative number can not have a real square root. So we need to extend the system of real numbers to a system in which we can find out the square roots of negative numbers. Euler (1707 - 1783) was the first mathematician to introduce the symbol i (iota) for positive square root of -1 i.e., $i = \sqrt{-1}$.

5.1.1 *Imaginary numbers*

Square root of a negative number is called an imaginary number., for example,

$$\sqrt{-9} = \sqrt{-1}\sqrt{9} = i3, \quad \sqrt{-7} = \sqrt{-1}\sqrt{7} = i\sqrt{7}$$

5.1.2 *Integral powers of i*

$i = \sqrt{-1}$, $i^2 = -1$, $i^3 = i^2 i = -i$, $i^4 = (i^2)^2 = (-1)^2 = 1$.

To compute i^n for $n > 4$, we divide n by 4 and write it in the form $n = 4m + r$, where m is quotient and r is remainder $(0 \le r \le 4)$

Hence $\quad i^n = i^{4m+r} = (i^4)^m \, . \, (i)^r = (1)^m \, (i)^r = i^r$

For example, $\quad (i)^{39} = i^{4 \times 9 + 3} = (i^4)^9 \, . \, (i)^3 = i^3 = -i$

and $\quad (i)^{-435} = i^{-(4 \times 108 + 3)} = (i)^{-(4 \times 108)} \, . \, (i)^{-3}$

$$= \frac{1}{(i^4)^{108}} \cdot \frac{1}{(i)^3} = \frac{i}{(i)^4} = i$$

(i) If a and b are positive real numbers, then

$$\sqrt{-a} \times \sqrt{-b} = \sqrt{-1}\sqrt{a} \times \sqrt{-1}\sqrt{b} = i\sqrt{a} \times i\sqrt{b} = -\sqrt{ab}$$

(ii) $\sqrt{a} \, . \, \sqrt{b} = \sqrt{ab}$ if a and b are positive or at least one of them is negative or zero. However, $\sqrt{a}\sqrt{b} \ne \sqrt{ab}$ if a and b, both are negative.

5.1.3 *Complex numbers*

(a) A number which can be written in the form $a + ib$, where a, b are real numbers and $i = \sqrt{-1}$ is called a complex number.

(b) If $z = a + ib$ is the complex number, then a and b are called real and imaginary parts, respectively, of the complex number and written as $\text{Re}(z) = a$, $\text{Im}(z) = b$.

(c) Order relations "greater than" and "less than" are not defined for complex numbers.

(d) If the imaginary part of a complex number is zero, then the complex number is known as purely real number and if real part is zero, then it is called purely imaginary number, for example, 2 is a purely real number because its imaginary part is zero and $3i$ is a purely imaginary number because its real part is zero.

5.1.4 *Algebra of complex numbers*

(a) Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are said to be equal if $a = c$ and $b = d$.

(b) Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers then $z_1 + z_2 = (a + c) + i\,(b + d)$.

5.1.5 *Addition of complex numbers satisfies the following properties*

1. As the sum of two complex numbers is again a complex number, the set of complex numbers is closed with respect to addition.
2. Addition of complex numbers is commutative, i.e., $z_1 + z_2 = z_2 + z_1$
3. Addition of complex numbers is associative, i.e., $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
4. For any complex number $z = x + i\,y$, there exist 0, i.e., $(0 + 0i)$ complex number such that $z + 0 = 0 + z = z$, known as identity element for addition.
5. For any complex number $z = x + iy$, there always exists a number $-z = -a - ib$ such that $z + (-z) = (-z) + z = 0$ and is known as the additive inverse of z.

5.1.6 *Multiplication of complex numbers*

Let $z_1 = a + ib$ and $z_2 = c + id$, be two complex numbers. Then

$z_1 \,.\, z_2 = (a + ib)\,(c + id) = (ac - bd) + i\,(ad + bc)$

1. As the product of two complex numbers is a complex number, the set of complex numbers is closed with respect to multiplication.
2. Multiplication of complex numbers is commutative, i.e., $z_1.z_2 = z_2.z_1$
3. Multiplication of complex numbers is associative, i.e., $(z_1.z_2)\,.\,z_3 = z_1\,.\,(z_2.z_3)$

4. For any complex number $z = x + iy$, there exists a complex number 1, i.e., $(1 + 0i)$ such that
   $z \, . \, 1 = 1 \, . \, z = z$, known as identity element for multiplication.
5. For any non zero complex number $z = x + i\,y$, there exists a complex number $\frac{1}{z}$
   such that $z \cdot \frac{1}{z} = \frac{1}{z} \cdot z = 1$, i.e., multiplicative inverse of $a + ib = \frac{1}{a+ib} = \frac{a-ib}{a^2+b^2}$.
6. For any three complex numbers z_1, z_2 and z_3,
   $$z_1 \, . \, (z_2 + z_3) = z_1 \, . \, z_2 + z_1 \, . \, z_3$$
   and $$(z_1 + z_2) \, . \, z_3 = z_1 \, . \, z_3 + z_2 \, . \, z_3$$
   i.e., for complex numbers multiplication is distributive over addition.

5.1.7 Let $z_1 = a + ib$ and $z_2 (\neq 0) = c + id$. Then

$$z_1 \div z_2 = \frac{z_1}{z_2} = \frac{a+ib}{c+id} = \frac{(ac+bd)}{c^2+d^2} + i\,\frac{(bc-ad)}{c^2+d^2}$$

5.1.8 *Conjugate of a complex number*

Let $z = a + ib$ be a complex number. Then a complex number obtained by changing the sign of imaginary part of the complex number is called the conjugate of z and it is denoted by $\bar{z}$, i.e., $\bar{z} = a - ib$.

Note that additive inverse of z is $-a - ib$ but conjugate of z is $a - ib$.

We have :

1. $\overline{(\bar{z})} = z$
2. $z + \bar{z} = 2\,\mathrm{Re}\,(z)$, $z - \bar{z} = 2\,i\,\mathrm{Im}(z)$
3. $z = \bar{z}$, if z is purely real.
4. $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary
5. $z \, . \, \bar{z} = \{\mathrm{Re}\,(z)\}^2 + \{\mathrm{Im}\,(z)\}^2$.
6. $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$, $\overline{(z_1 - z_2)} = \bar{z}_1 - \bar{z}_2$
7. $\overline{(z_1 \, . \, z_2)} = (\bar{z}_1)\,(\bar{z}_2)$, $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{(\bar{z}_1)}{(\bar{z}_2)}$ $(\bar{z}_2 \neq 0)$

5.1.9 *Modulus of a complex number*

Let $z = a + ib$ be a complex number. Then the positive square root of the sum of square of real part and square of imaginary part is called modulus (absolute value) of z and it is denoted by $|z|$ i.e., $|z| = \sqrt{a^2+b^2}$

In the set of complex numbers $z_1 > z_2$ or $z_1 < z_2$ are meaningless but

$$|z_1| > |z_2| \text{ or } |z_1| < |z_2|$$

are meaningful because $|z_1|$ and $|z_2|$ are real numbers.

5.1.10 *Properties of modulus of a complex number*

1. $|z| = 0 \Leftrightarrow z = 0$ i.e., Re $(z) = 0$ and Im $(z) = 0$
2. $|z| = |\bar{z}| = |-z|$
3. $-|z| \leq \text{Re}(z) \leq |z|$ and $-|z| \leq \text{Im}(z) \leq |z|$
4. $z\,\bar{z} = |z|^2,\ |z^2| = |\bar{z}|^2$
5. $|z_1 z_2| = |z_1|.|z_2|,\ \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}\ (z_2 \neq 0)$
6. $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1 \bar{z}_2)$
7. $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\text{Re}(z_1 \bar{z}_2)$
8. $|z_1 + z_2| \leq |z_1| + |z_2|$
9. $|z_1 - z_2| \geq |z_1| - |z_2|$
10. $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$

    In particular:

    $|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2(|z_1|^2 + |z_2|^2)$
11. As stated earlier multiplicative inverse (reciprocal) of a complex number $z = a + ib\ (\neq 0)$ is

$$\frac{1}{z} = \frac{a - ib}{a^2 + b^2} = \frac{\bar{z}}{|z|^2}$$

5.2 Argand Plane

A complex number $z = a + ib$ can be represented by a unique point P (a, b) in the cartesian plane referred to a pair of rectangular axes. The complex number $0 + 0i$ represent the origin 0 (0, 0). A purely real number a, i.e., $(a + 0i)$ is represented by the point $(a, 0)$ on x - axis. Therefore, x-axis is called real axis. A purely imaginary number

ib, i.e., $(0 + ib)$ is represented by the point $(0, b)$ on y-axis. Therefore, y-axis is called imaginary axis.

Similarly, the representation of complex numbers as points in the plane is known as **Argand diagram**. The plane representing complex numbers as points is called complex plane or Argand plane or Gaussian plane.

If two complex numbers z_1 and z_2 be represented by the points P and Q in the complex plane, then

$$|z_1 - z_2| = \text{PQ}$$

5.2.1 *Polar form of a complex number*

Let P be a point representing a non-zero complex number $z = a + ib$ in the Argand plane. If OP makes an angle θ with the positive direction of x-axis, then $z = r(\cos\theta + i\sin\theta)$ is called the polar form of the complex number, where $r = |z| = \sqrt{a^2 + b^2}$ and $\tan\theta = \dfrac{b}{a}$. Here θ is called argument or amplitude of z and we write it as $\arg(z) = \theta$.

The unique value of θ such that $-\pi \leq \theta \leq \pi$ is called the principal argument.

$$\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

5.2.2 *Solution of a quadratic equation*

The equations $ax^2 + bx + c = 0$, where a, b and c are numbers (real or complex, $a \neq 0$) is called the general quadratic equation in variable x. The values of the variable satisfying the given equation are called roots of the equation.

The quadratic equation $ax^2 + bx + c = 0$ with real coefficients has two roots given by $\dfrac{-b+\sqrt{\text{D}}}{2a}$ and $\dfrac{-b-\sqrt{\text{D}}}{2a}$, where $\text{D} = b^2 - 4ac$, called the discriminant of the equation.

Notes

1. When D = 0, roots of the quadratic equation are real and equal. When D > 0, roots are real and unequal.
   Further, if $a, b, c \in \mathbf{Q}$ and D is a perfect square, then the roots of the equation are rational and unequal, and if $a, b, c \in \mathbf{Q}$ and D is not a perfect square, then the roots are irrational and occur in pair.

When D < 0, roots of the quadratic equation are non real (or complex).

2. Let α, β be the roots of the quadratic equation $ax^2 + bx + c = 0$, then sum of the roots

   $(\alpha + \beta) = \frac{-b}{a}$ and the product of the roots $(\alpha . \beta) = \frac{c}{a}$.

3. Let S and P be the sum of roots and product of roots, respectively, of a quadratic equation. Then the quadratic equation is given by $x^2 - Sx + P = 0$.

5.2 Solved Exmaples

Short Answer Type

Example 1 Evaluate : $(1 + i)^6 + (1 - i)^3$

Solution $(1 + i)^6 = \{(1 + i)^2\}^3 = (1 + i^2 + 2i)^3 = (1 - 1 + 2i)^3 = 8\, i^3 = -8i$

and $(1 - i)^3 = 1 - i^3 - 3i + 3i^2 = 1 + i - 3i - 3 = -2 - 2i$

Therefore, $(1 + i)^6 + (1 - i)^3 = -8i - 2 - 2i = -2 - 10i$

Example 2 If $(x+iy)^{\frac{1}{3}} = a + ib$, where $x, y, a, b \in$ R, show that $\frac{x}{a} - \frac{y}{b} = -2\,(a^2 + b^2)$

Solution $(x+iy)^{\frac{1}{3}} = a + ib$

$\Rightarrow \quad x + iy = (a + ib)^3$

i.e., $\quad x + iy = a^3 + i^3 b^3 + 3iab\,(a + ib)$

$= a^3 - ib^3 + i3a^2b - 3ab^2$

$= a^3 - 3ab^2 + i\,(3a^2b - b^3)$

$\Rightarrow \quad x = a^3 - 3ab^2$ and $y = 3a^2b - b^3$

Thus $\quad \frac{x}{a} = a^2 - 3b^2$ and $\frac{y}{b} = 3a^2 - b^2$

So, $\quad \frac{x}{a} - \frac{y}{b} = a^2 - 3b^2 - 3a^2 + b^2 = -2\,a^2 - 2b^2 = -2\,(a^2 + b^2)$.

Example 3 Solve the equation $z^2 = \bar{z}$, where $z = x + iy$

Solution $z^2 = \bar{z} \quad \Rightarrow \quad x^2 - y^2 + i2xy = x - iy$

Therefore, $x^2 - y^2 = x \quad \ldots (1) \quad$ and $\quad 2xy = -y \quad \ldots (2)$

From (2), we have $y = 0$ or $x = -\frac{1}{2}$

When $y = 0$, from (1), we get $x^2 - x = 0$, i.e., $x = 0$ or $x = 1$.

When $x = -\frac{1}{2}$, from (1), we get $y^2 = \frac{1}{4} + \frac{1}{2}$ or $y^2 = \frac{3}{4}$, i.e., $y = \pm\frac{\sqrt{3}}{2}$.

Hence, the solutions of the given equation are

$$0 + i0,\ 1 + i0,\ -\frac{1}{2} + i\,\frac{\sqrt{3}}{2},\ -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Example 4 If the imaginary part of $\frac{2z+1}{iz+1}$ is -2, then show that the locus of the point representing z in the argand plane is a straight line.

Solution Let $z = x + iy$. Then

$$\frac{2z+1}{iz+1} = \frac{2(x+iy)+1}{i(x+iy)+1} = \frac{(2x+1)+i2y}{(1-y)+ix}$$

$$= \frac{\{(2x+1)+i2y\}}{\{(1-y)+ix\}} \times \frac{\{(1-y)-ix\}}{\{(1-y)-ix\}}$$

$$= \frac{(2x+1-y)+i(2y-2y^2-2x^2-x)}{1+y^2-2y+x^2}$$

Thus $\quad \text{Im}\left(\frac{2z+1}{iz+1}\right) = \frac{2y-2y^2-2x^2-x}{1+y^2-2y+x^2}$

But $\quad \text{Im}\left(\frac{2z+1}{iz+1}\right) = -2 \quad$ (Given)

So $\quad \frac{2y-2y^2-2x^2-x}{1+y^2-2y+x^2} = -2$

$\Rightarrow \quad 2y - 2y^2 - 2x^2 - x = -2 - 2y^2 + 4y - 2x^2$

i.e., $\quad x + 2y - 2 = 0$, which is the equation of a line.

Example 5 If $\left|z^2 - 1\right| = \left|z\right|^2 + 1$, then show that z lies on imaginary axis.

Solution Let $z = x + iy$. Then $|z^2 - 1| = |z|^2 + 1$

$\Rightarrow \quad \left|x^2 - y^2 - 1 + i\,2xy\right| = \left|x + iy\right|^2 + 1$

$\Rightarrow \quad (x^2 - y^2 - 1)^2 + 4x^2y^2 = (x^2 + y^2 + 1)^2$

$\Rightarrow \quad 4x^2 = 0 \quad$ i.e., $\quad x = 0$

Hence z lies on y-axis.

Example 6 Let z_1 and z_2 be two complex numbers such that $\bar{z}_1 + i\bar{z}_2 = 0$ and $\arg(z_1 z_2) = \pi$. Then find $\arg(z_1)$.

Solution Given that $\bar{z}_1 + i\bar{z}_2 = 0$

$\Rightarrow \quad z_1 = iz_2$, i.e., $z_2 = -iz_1$

Thus $\quad \arg(z_1 z_2) = \arg z_1 + \arg(-iz_1) = \pi$

$\Rightarrow \quad \arg(-iz_1^2) = \pi$

$\Rightarrow \quad \arg(-i) + \arg(z_1^2) = \pi$

$\Rightarrow \quad \arg(-i) + 2\arg(z_1) = \pi$

$\Rightarrow \quad \dfrac{-\pi}{2} + 2\arg(z_1) = \pi$

$\Rightarrow \quad \arg(z_1) = \dfrac{3\pi}{4}$

Example 7 Let z_1 and z_2 be two complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$.

Then show that $\arg(z_1) - \arg(z_2) = 0$.

Solution Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

where $\quad r_1 = |z_1|$, $\arg(z_1) = \theta_1$, $r_2 = |z_2|$, $\arg(z_2) = \theta_2$.

We have, $\quad |z_1 + z_2| = |z_1| + |z_2|$

$= \left|r_1(\cos\theta_1 + \cos\theta_2) + r_2(\cos\theta_2 + \sin\theta_2)\right| = r_1 + r_2$

$= r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2) = (r_1 + r_2)^2 \Rightarrow \cos(\theta_1 - \theta_2) = 1$

$\Rightarrow \theta_1 - \theta_2$ i.e. $\arg z_1 = \arg z_2$

Example 8 If z_1, z_2, z_3 are complex numbers such that

$|z_1| = |z_2| = |z_3| = \left|\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3}\right| = 1$, then find the value of $|z_1 + z_2 + z_3|$.

Solution $|z_1| = |z_2| = |z_3| = 1$

$\Rightarrow \quad |z_1|^2 = |z_2|^2 = |z_3|^2 = 1$

$\Rightarrow \quad z_1\,\overline{z}_1 = z_2\,\overline{z}_2 = z_3\,\overline{z}_3 = 1$

$\Rightarrow \quad \overline{z}_1 = \dfrac{1}{z_1},\ \overline{z}_2 = \dfrac{1}{z_2},\ \overline{z}_3 = \dfrac{1}{z_3}$

Given that $\left|\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3}\right| = 1$

$\Rightarrow \quad |\overline{z}_1 + \overline{z}_2 + \overline{z}_3| = 1$, i.e., $|\overline{z_1 + z_2 + z_3}| = 1$

$\Rightarrow \quad |z_1 + z_2 + z_3| = 1$

Example 9 If a complex number z lies in the interior or on the boundary of a circle of radius 3 units and centre (– 4, 0), find the greatest and least values of $|z+1|$.

Solution Distance of the point representing z from the centre of the circle is $|z-(-4+i0)| = |z+4|$.

According to given condition $|z+4| \leq 3$.

Now $|z+1| = |z+4-3| \leq |z+4| + |-3| \leq 3+3 = 6$

Therefore, greatest value of $|z+1|$ is 6.

Since least value of the modulus of a complex number is zero, the least value of $|z+1| = 0$.

Example 10 Locate the points for which $3 < |z| < 4$

Solution $|z| < 4 \Rightarrow x^2 + y^2 < 16$ which is the interior of circle with centre at origin and radius 4 units, and $|z| > 3 \Rightarrow x^2 + y^2 > 9$ which is exterior of circle with centre at origin and radius 3 units. Hence $3 < |z| < 4$ is the portion between two circles $x^2 + y^2 = 9$ and $x^2 + y^2 = 16$.

Example 11 Find the value of $2x^4 + 5x^3 + 7x^2 - x + 41$, when $x = -2 - \sqrt{3}\,i$

Solution $x + 2 = -\sqrt{3}\,i \Rightarrow x^2 + 4x + 7 = 0$

Therefore $\quad 2x^4 + 5x^3 + 7x^2 - x + 41 = (x^2 + 4x + 7)(2x^2 - 3x + 5) + 6$

$= 0 \times (2x^2 - 3x + 5) + 6 = 6.$

Example 12 Find the value of P such that the difference of the roots of the equation $x^2 - Px + 8 = 0$ is 2.

Solution Let α, β be the roots of the equation $x^2 - Px + 8 = 0$

Therefore $\alpha + \beta = P$ and $\alpha . \beta = 8$.

Now $\alpha - \beta = \pm\sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$

Therefore $2 = \pm\sqrt{P^2 - 32}$

$\Rightarrow$ $P^2 - 32 = 4$, i.e., $P = \pm 6$.

Example 13 Find the value of a such that the sum of the squares of the roots of the equation $x^2 - (a - 2)x - (a + 1) = 0$ is least.

Solution Let α, β be the roots of the equation

Therefore, $\alpha + \beta = a - 2$ and $\alpha\beta = -(a + 1)$

Now
$$\begin{aligned}\alpha^2 + \beta^2 &= (\alpha + \beta)^2 - 2\alpha\beta \\ &= (a - 2)^2 + 2(a + 1) \\ &= (a - 1)^2 + 5\end{aligned}$$

Therefore, $\alpha^2 + \beta^2$ will be minimum if $(a - 1)^2 = 0$, i.e., $a = 1$.

Long Answer Type

Example 14 Find the value of k if for the complex numbers z_1 and z_2,

$$|1 - \bar{z}_1 z_2|^2 - |z_1 - z_2|^2 = k(1 - |z_1|^2)(1 - |z_2|^2)$$

Solution

$$\begin{aligned}\text{L.H.S.} &= |1 - \bar{z}_1 z_2|^2 - |z_1 - z_2|^2 \\ &= (1 - \bar{z}_1 z_2)\,\overline{(1 - \bar{z}_1 z_2)} - (z_1 - z_2)\,\overline{(z_1 - z_2)} \\ &= (1 - \bar{z}_1 z_2)\,(1 - z_1 \bar{z}_2) - (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \\ &= 1 + z_1 \bar{z}_1\, z_2 \bar{z}_2 - z_1 \bar{z}_1 - z_2 \bar{z}_2 \\ &= 1 + |z_1|^2 \cdot |z_2|^2 - |z_1|^2 - |z_2|^2 \\ &= (1 - |z_1|^2)\,(1 - |z_2|^2)\end{aligned}$$

$$\text{R.H.S.} = k\,(1 - |z_1|^2)\,(1 - |z_2|^2)$$

$\Rightarrow$ $k = 1$

Hence, equating LHS and RHS, we get $k = 1$.

Example 15 If z_1 and z_2 both satisfy $z + \bar{z} = 2|z-1|$ arg $(z_1 - z_2) = \frac{\pi}{4}$, then find I*m* $(z_1 + z_2)$.

Solution Let $z = x + iy$, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

Then $\quad z + \bar{z} = 2|z-1|$

$\Rightarrow \quad (x + iy) + (x - iy) = 2\,|x-1+iy|$

$\Rightarrow \quad 2x = 1 + y^2 \qquad \text{... (1)}$

Since z_1 and z_2 both satisfy (1), we have

$2x_1 = 1 + y_1^2$ ... and $2x_2 = 1 + y_2^2$

$\Rightarrow \quad 2\,(x_1 - x_2) = (y_1 + y_2)\,(y_1 - y_2)$

$\Rightarrow \quad 2 = (y_1 + y_2)\left(\frac{y_1 - y_2}{x_1 - x_2}\right) \qquad \text{... (2)}$

Again $\quad z_1 - z_2 = (x_1 - x_2) + i\,(y_1 - y_2)$

Therefore, $\quad \tan\theta = \frac{y_1 - y_2}{x_1 - x_2}$, where $\theta = \arg(z_1 - z_2)$

$\Rightarrow \quad \tan\frac{\pi}{4} = \frac{y_1 - y_2}{x_1 - x_2} \qquad \left(\text{since } \theta = \frac{\pi}{4}\right)$

i.e., $\quad 1 = \frac{y_1 - y_2}{x_1 - x_2}$

From (2), we get $2 = y_1 + y_2$, i.e., Im $(z_1 + z_2) = 2$

Objective Type Questions

Example 16 Fill in the blanks:

(i) The real value of 'a' for which $3i^3 - 2ai^2 + (1 - a)i + 5$ is real is ________.

(ii) If $|z| = 2$ and arg $(z) = \frac{\pi}{4}$, then z = ________.

(iii) The locus of z satisfying arg $(z) = \frac{\pi}{3}$ is _______.

(iv) The value of $(-\sqrt{-1})^{4n-3}$, where $n \in \mathbf{N}$, is ______.

(v) The conjugate of the complex number $\frac{1-i}{1+i}$ is _____.

(vi) If a complex number lies in the third quadrant, then its conjugate lies in the ______.

(vii) If $(2 + i)(2 + 2i)(2 + 3i) \dots (2 + ni) = x + iy$, then $5.8.13 \dots (4 + n^2) =$ ______.

Solution

(i) $3i^3 - 2ai^2 + (1 - a)i + 5 = -3i + 2a + 5 + (1 - a)i$

$= 2a + 5 + (-a - 2)i$, which is real if $-a - 2 = 0$ i.e. $a = -2$.

(ii) $z = |z|\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}(1+i)$

(iii) Let $z = x + iy$. Then its polar form is $z = r(\cos\theta + i\sin\theta)$, where $\tan\theta = \frac{y}{x}$ and θ is arg (z). Given that $\theta = \frac{\pi}{3}$. Thus.

$\tan\frac{\pi}{3} = \frac{y}{x} \Rightarrow y = \sqrt{3}x$, where $x > 0$, $y > 0$.

Hence, locus of z is the part of $y = \sqrt{3}x$ in the first quadrant except origin.

(iv) Here $(-\sqrt{-1})^{4n-3} = (-i)^{4n-3} = (-i)^{4n}(-i)^{-3} = \frac{1}{(-i)^3}$

$$= \frac{1}{-i^3} = \frac{1}{i} = \frac{i}{i^2} = -i$$

(v) $\frac{1-i}{1+i} = \frac{1-i}{1+i} \times \frac{1-i}{1-i} = \frac{1+i^2-2i}{1-i^2} = \frac{1-1-2i}{1+1} = -i$

Hence, conjugate of $\frac{1-i}{1+i}$ is i.

(vi) Conjugate of a complex number is the image of the complex number about the x-axis. Therefore, if a number lies in the third quadrant, then its image lies in the second quadrant.

(vii) Given that $(2 + i)(2 + 2i)(2 + 3i) \dots (2 + ni) = x + iy$ ... (1)

$\Rightarrow \quad \overline{(2+i)}\,\overline{(2+2i)}\,\overline{(2+3i)} \dots \overline{(2+ni)} = \left(\overline{x+iy}\right) = (x-iy)$

i.e., $\quad (2 - i)(2 - 2i)(2 - 3i) \dots (2 - ni) = x - iy$ ... (2)

Multiplying (1) and (2), we get 5.8.13 ... $(4 + n^2) = x^2 + y^2$.

Example 17 State true or false for the following:

(i) Multiplication of a non-zero complex number by i rotates it through a right angle in the anti- clockwise direction.

(ii) The complex number $\cos\theta + i \sin\theta$ can be zero for some θ.

(iii) If a complex number coincides with its conjugate, then the number must lie on imaginary axis.

(iv) The argument of the complex number $z = (1 + i\sqrt{3})(1 + i)(\cos\theta + i \sin\theta)$ is $\frac{7\pi}{12} + \theta$

(v) The points representing the complex number z for which $|z+1| < |z-1|$ lies in the interior of a circle.

(vi) If three complex numbers z_1, z_2 and z_3 are in A.P., then they lie on a circle in the complex plane.

(vii) If n is a positive integer, then the value of $i^n + (i)^{n+1} + (i)^{n+2} + (i)^{n+3}$ is 0.

Solution

(i) True. Let $z = 2 + 3i$ be complex number represented by OP. Then $iz = -3 + 2i$, represented by OQ, where if OP is rotated in the anticlockwise direction through a right angle, it coincides with OQ.

(ii) False. Because $\cos\theta + i\sin\theta = 0 \Rightarrow \cos\theta = 0$ and $\sin\theta = 0$. But there is no value of θ for which $\cos\theta$ and $\sin\theta$ both are zero.

(iii) False, because $x + iy = x - iy \Rightarrow y = 0 \Rightarrow$ number lies on x-axis.

(iv) True, $\arg(z) = \arg(1 + i\sqrt{3}) + \arg(1 + i) + \arg(\cos\theta + i\sin\theta)$

$$\frac{\pi}{3} + \frac{\pi}{4} + \theta = \frac{7\pi}{12} + \theta$$

(v) False, because $|x+iy+1| < |x+iy-1|$

$\Rightarrow \quad (x + 1)^2 + y^2 < (x - 1)^2 + y^2$ which gives $4x < 0$.

(vi) False, because if z_1, z_2 and z_3 are in A.P., then $z_2 = \frac{z_1 + z_3}{2} \Rightarrow z_2$ is the midpoint of z_1 and z_3, which implies that the points z_1, z_2, z_3 are collinear.

(vii) True, because $i^n + (i)^{n+1} + (i)^{n+2} + (i)^{n+3}$

$$= i^n (1 + i + i^2 + i^3) = i^n (1 + i - 1 - i)$$

$$= i^n (0) = 0$$

Example 18 Match the statements of column A and B.

	Column A		Column B
(a)	The value of $1+i^2+i^4+i^6+\ldots i^{20}$ is	(i)	purely imaginary complex number
(b)	The value of i^{-1097} is	(ii)	purely real complex number
(c)	Conjugate of $1+i$ lies in	(iii)	second quadrant
(d)	$\frac{1+2i}{1-i}$ lies in	(iv)	Fourth quadrant
(e)	If $a, b, c \in$ R and $b^2-4ac<0$, then the roots of the equation $ax^2+bx+c=0$ are non real (complex) and	(v)	may not occur in conjugate pairs
(f)	If $a, b, c \in$ R and $b^2-4ac>0$, and b^2-4ac is a perfect square, then the roots of the equation $ax^2+bx+c=0$	(vi)	may occur in conjugate pairs

Solution

(a) $\Leftrightarrow$ (ii), because $1+i^2+i^4+i^6+\ldots+i^{20}$
$=1-1+1-1+\ldots+1=1$ (which is purely a real complex number)

(b) $\Leftrightarrow$ (i), because $i^{-1097}=\frac{1}{(i)^{1097}}=\frac{1}{i^{4\times 274+1}}=\frac{1}{\{(i)^4\}^{274}(i)}=\frac{1}{i}=\frac{i}{i^2}=-i$

which is purely imaginary complex number.

(c) $\Leftrightarrow$ (iv), conjugate of $1+i$ is $1-i$, which is represented by the point $(1,-1)$ in the fourth quadrant.

(d) $\Leftrightarrow$ (iii), because $\frac{1+2i}{1-i}=\frac{1+2i}{1-i}\times\frac{1+i}{1+i}=\frac{-1+3i}{2}=-\frac{1}{2}+\frac{3}{2}i$, which is represented by the point $\left(-\frac{1}{2},\frac{3}{2}\right)$ in the second quadrant.

(e) $\Leftrightarrow$ (vi), If $b^2-4ac<0=\text{D}<0$, i.e., square root of D is a imaginary number, therefore, roots are $x=\frac{-b\pm\text{Imaginary Number}}{2a}$, i.e., roots are in conjugate pairs.